

Sketches from the life of hypercomplex numbers.

K.V. Andreev

Every science result represents an element of a puzzle from the general picture of the world. Every element has connectors with the help of which it connected to its neighbors. A sketch is called a description of how these connectors are connected to each other. But each painter represents the world as he sees it. Therefore, all pictures of the world are different. And all elements of a puzzle are connected to each other whimsically. This article is a description of the small piece of a full picture from the hypercomplex life as it is seen to the author. Therefore, in the article, the main ideas of the induction construction arXiv:1204.0194, arXiv:1110.4737, arXiv:1202.0941, arXiv:1208.4466 are considered in the form of sketches. By and large, the article establishes a link between Clifford algebras and alternative-elastic algebras at the level of connectors.

1. Real numbers.

The first axiom system for arithmetic of real numbers was published by Hilbert in 1900. Real numbers can be determined by the following axioms:

I. Field Axioms.

- I-1. Closure of \mathbb{R} under addition: $\forall a, b \in \mathbb{R}, \exists! a + b =: c \in \mathbb{R}$.
- I-2. Associativity of addition: $\forall a, b, c \in \mathbb{R}, a + (b + c) = (a + b) + c$.
- I-3. Commutativity of addition: $\forall a, b \in \mathbb{R}, a + b = b + a$.
- I-4. Existence of additive identity element: $\forall a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$.
- I-5. Existence of additive inverse: $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}, a + b = 0$.
- I-6. Closure of \mathbb{R} under multiplication: $\forall a, b \in \mathbb{R}, \exists! a \cdot b =: c \in \mathbb{R}$.
- I-7. Associativity of multiplication: $\forall a, b, c \in \mathbb{R} a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- I-8. Commutativity of multiplication: $\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$.
- I-9. Existence of multiplicative identity element: $\forall a \in \mathbb{R}, \exists 1 \in \mathbb{R}, a \cdot 1 = a$.
- I-10. Existence of multiplicative inverse: $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}, a \cdot b = 1$.
- I-11. Distributivity of multiplication over addition: $\forall a, b, c \in \mathbb{R} a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
- I-12. Nontriviality: $1 \neq 0$.

II. Ordering Axioms.

- II-1. Reflexivity: $\forall a \in \mathbb{R}, a \leq a$.
- II-2. Antisymmetry: $\forall a, b \in \mathbb{R}, (a \leq b) \wedge (b \leq a) \Rightarrow (a = b)$.
- II-3. Transitivity: $\forall a, b, c \in \mathbb{R}, (a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c)$.
- II-4. Totality: $\forall a, b \in \mathbb{R}, (a \leq b) \vee (b \leq a)$.
- II-5. Relationship between order and addition: $\forall a, b, c \in \mathbb{R}, (a \leq b) \Rightarrow (a + c \leq b + c)$.
- II-6. Relationship between order and multiplication: $\forall a, b, c \in \mathbb{R}, (0 \leq a) \wedge (0 \leq b) \Rightarrow (0 \leq ab)$.

III. Completeness Axiom.

If A and B are nonempty subsets of \mathbb{R} such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

Note that in 1936, Alfred Tarski set out an axiomatization of the real numbers, consisting of only the 9 axioms [34, p. 275 (rus)].

2. Complex numbers.

Complex numbers were introduced by Italian mathematician Gerolamo Cardano in the 16th century (1545). A complex number is represented as $a + bi$, where i is called the imaginary unit and $i^2 = -1$. a and b are real numbers. Unlike real numbers, complex numbers don't form an ordered field. The complex number $a - bi$ is called *complex conjugate* to $a + bi$.

3. Cayley-Dickson numbers.

Let an algebra \mathbb{A} be given then the algebra \mathbb{A}^2 is determined as follows from [1] (in 1942). For all $a, b \in \mathbb{A}$ and the new imaginary unit i , the hypercomplex numbers $x := a + bi$ form \mathbb{A}^2 and the parities

1. $a(bi) = (ba)i$;
2. $a(ib) = (a\bar{b})i$;
3. $(ai)(bi) = -\bar{b}a$;
4. $\overline{a + bi} = \bar{a} - bi$

are executed.

4. Quaternions.

Quaternions were introduced by Hamilton in 1843 [17] with the three imaginary units ($ij = k$, $jk = i$, $ki = j$). However, the quaternions form a non-commutative associative division algebra. Each quaternion is formed by a pair of complex numbers according to the Cayley-Dickson procedure.

5. Octonions.

Octonions were introduced by John T. Graves (in 1843) and Arthur Cayley (in 1845) independently [14], [11] with seven imaginary units. However, the octonions form a non-commutative non-associative alternative division algebra. Each octonion is formed by a pair of quaternions according to the Cayley-Dickson procedure. Instead of the associative law, the alternative laws exist: $(aa)b = a(ab)$ (left), $b(aa) = (ba)a$ (right). The automorphism group has the dimension equal to 14 and is a subgroup of the orthogonal group whose dimension is equal to 28. One of the invariants is the algebra identity that reduces the dimension on 7. What is the second invariant that reduces the dimension on 7 else?

6. Sedenions.

Sedenions were introduced by John T. Graves with 15 imaginary units. However, the sedenions form an non-commutative non-associative alternative-elastic non-division algebra. Each sedenion is formed by a pair of octonions according to the Cayley-Dickson procedure. Instead of the alternative law, the weakly alternative law exists: $(aa)b - a(ab) = b(aa) - (ba)a$. This identity is equivalent to both the flexible law $a(ba) = (ab)a$ and the power-associative law $a(aa)=(aa)a$ [1], [19], [6].

7. Alternative-elastic algebras.

An alternative-elastic algebra \mathbb{A}^n over field \mathbb{R} is formed by the following axioms [6]:

1. Closure of \mathbb{A}^n under addition: $\forall a, b \in \mathbb{A}^n, \exists! a + b =: c \in \mathbb{A}^n$.
2. Associativity of addition: $\forall a, b, c \in \mathbb{A}^n, a + (b + c) = (a + b) + c$.
3. Commutativity of addition: $\forall a, b \in \mathbb{A}^n, a + b = b + a$.
4. Existence of additive identity element: $\forall a \in \mathbb{A}^n, \exists 0 \in \mathbb{A}^n, a + 0 = a$.
5. Existence of additive inverse: $\forall a \in \mathbb{A}^n, \exists b \in \mathbb{A}^n, a + b = 0$.
6. Closure of \mathbb{A}^n under multiplication: $\forall a, b \in \mathbb{A}^n, \exists! a \cdot b =: c \in \mathbb{A}^n$.
7. Weakly alternativity of multiplication: $\forall a, b \in \mathbb{A}^n (a \cdot a) \cdot b - a \cdot (a \cdot b) = b \cdot (a \cdot a) - (b \cdot a) \cdot a$.
8. Existence of multiplicative identity element: $\forall a \in \mathbb{A}^n, \exists 1 \in \mathbb{A}^n, a \cdot 1 = a$.
9. Existence of multiplicative inverse: $\forall a \in \mathbb{A}^n, \exists b \in \mathbb{A}^n, a \cdot b = 1$.
10. Distributivity of multiplication over addition: $\forall a, b, c \in \mathbb{A}^n a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
11. Nontriviality: $1 \neq 0$.

One can determine the metric on the algebra as $\langle a, b \rangle_e := \frac{1}{2}(a\bar{b} + b\bar{a})$. In particular, each Cayley-Dickson algebra, constructed from \mathbb{R} by Cayley-Dickson procedure applied a number of times, is a metric alternative-elastic algebra.

8. Complex and real representations. Real inclusion.

In the article [24] (in 1959), the Norden affinor was introduced as $\Delta_{\tilde{\Lambda}}^{\tilde{\Psi}} := \frac{1}{2}(\delta_{\tilde{\Lambda}}^{\tilde{\Psi}} + i f_{\tilde{\Lambda}}^{\tilde{\Psi}})$ ($\tilde{\Lambda}, \tilde{\Psi}, \dots = \overline{1, 2n}$), where $f_{\tilde{\Lambda}}^{\tilde{\Psi}}$ is a complex structure. In 1989, this allowed to introduce the Neifeld operators [22] $\Delta_{\tilde{\Lambda}}^{\tilde{\Psi}} := m_{\tilde{\Lambda}}^{\Lambda} m_{\Lambda}^{\tilde{\Psi}}$ ($\Lambda, \Psi, \dots = \overline{1, n}$). This provided a transition between the real and complex representations: $\mathbb{R}_{(n,n)}^{2n}$ and \mathbb{C}^n . In addition, one can locally define a real inclusion with the help of an inclusion operator $H_i^{\Lambda}: \mathbb{R}^n \rightarrow \mathbb{C}^n$ according to [22] and can determine the involution in the complex space as $S_{\Lambda}^{\Psi'} = H_{\Lambda}^i \bar{H}_i^{\Psi'}$ ($\bar{H}_i^{\Psi'} := \overline{H_i^{\Psi}}$ - complex conjugation) ($i, j, \dots = \overline{1, n}$). In particular, for the metric tensors, the connecting parities $g_{\Lambda\Psi} := G_{\tilde{\Lambda}\tilde{\Psi}} m_{\Lambda}^{\tilde{\Lambda}} m_{\Psi}^{\tilde{\Psi}}, \bar{g}_{\Lambda'\Psi'} := G_{\tilde{\Lambda}\tilde{\Psi}} \bar{m}_{\Lambda'}^{\tilde{\Lambda}} \bar{m}_{\Psi'}^{\tilde{\Psi}}, g_{ij} := H_i^{\Lambda} H_j^{\Psi} g_{\Lambda\Psi} = \bar{H}_i^{\Lambda'} \bar{H}_j^{\Psi'} \bar{g}_{\Lambda'\Psi'}$ are executed [2, p. 12, eq. (3.11); p. 17, eq. (4.10)], [3, p. 11, eq. (3.11); p. 15, eq. (4.10)(eng), p. 135(12), eq. (3.11); p. 140(17), eq. (4.10)(rus)].

9. Clifford algebras.

Clifford algebra (was discovered in 1878 by Clifford William Kingdom) $CL(G_{(n,n)}^{2n})$ over the field \mathbb{R} is determined by generators $\gamma_{\tilde{A}}$ ($\tilde{A}, \tilde{\Psi}, \dots = \overline{1, 2n}$) satisfied the Clifford equation $\gamma_{\tilde{A}}\gamma_{\tilde{\Psi}} + \gamma_{\tilde{\Psi}}\gamma_{\tilde{A}} = G_{\tilde{A}\tilde{\Psi}}E$, where E is identity operator. Each generator is represented by a matrix whose the dimension is equal to $(2N)^2 \times (2N)^2$, where $N := 2^{n/2-1}$. The space $\mathbb{R}_{(n,n)}^{2n}(\mathbb{C}^n)$, on which the generators are defined, is called *base space*. One can pass to one of the $2N$ complex representations such the generators (see next Section) that leads to the complex algebra $CL(g^n)$, each generator of which is represented by a matrix whose the dimension is equal to $2N \times 2N$. A generator of the Clifford algebra $CL(g_{(n,0)}^n)$ can be constructed with the help of an inclusion operator for the inclusion of the real space \mathbb{R}^n into the complex space \mathbb{C}^n . Note that the Clifford generators can be reduced by the formula $\gamma_{\Lambda} = \begin{pmatrix} 0 & \sigma_{\Lambda} \\ \eta_{\Lambda} & 0 \end{pmatrix}$.

And the components can be formalized as $\eta_{\Lambda} := \eta_{\Lambda}^{AB}$, $\sigma_{\Lambda} := \sigma_{\Lambda AB} = \eta_{\Lambda BA}$; ($A, B, \dots = \overline{1, N}$, $\Lambda, \Psi, \dots = \overline{1, n}$). And the Clifford equation will take the form $\eta_{\Lambda}^{AB}\eta_{\Psi CB} + \eta_{\Psi}^{AB}\eta_{\Lambda CB} = g_{\Lambda\Psi}\delta_C^A$. Accordingly, the space \mathbb{C}^N (\mathbb{C}^{2N^2}) is called *spinor space*. The operators η_{Λ}^{AB} are the connecting ones and define *spinor formalism*. This means that a path of the algebraic load is transferred onto the connecting operators; this simplifies some tensor parities and leads to new spinor ones, which are not obvious in the tensor form. A good example is the spinor classification of the Weil tensor [25, v. 2, p. 256(eng)] unlike the tensor ones [26]. For our purposes, such an algebraic expression, which carries most of the algebraic load, will be the triple product $\eta_{\Lambda}^{AB}\eta_{\Psi AC}\eta_{\Phi DB}$ [21, p. 272] written down in one form or another. Note that both the connecting operators η_{Λ}^{AB} and the structural constants of an algebra $\eta_{\Lambda\Psi}^{\Phi}$ have three indexes for each. Can one associate these two objects together? And what will be the third object of this link? Will this be the second invariant for the octonion automorphism group that reduces the dimension on 7 else? The answer to these questions is here [5], [3, p. 48, eq. (9.3)(eng), p. 175(55), eq. (9.3)(rus)], [2, p. 55, eq. (9.3)], [6, eq. (43)-(45)], [7, eq. (2)].

$$\eta_{\Lambda\Psi}^{\Phi} := \sqrt{2}\eta_{\Lambda}^{AB}\eta_{\Psi CA}\eta_{\Phi DB}\theta^{CD},$$

where θ^{CD} is the controlling symmetric spin-tensor and is the second invariant for the octonion automorphism group. For the real inclusion $H_i^{\Lambda} : \mathbb{R}^n \rightarrow \mathbb{C}^n$ ($i, j, \dots = \overline{1, n}$), such an algebra with the structural constant η_{ij}^k will be a metric alternative-elastic algebra over the field \mathbb{R} . Well, but how to solve the Clifford equation for even n ? And why controlling symmetric spin-tensor is an invariant under the octonion automorphism group? We need to look at the history again. But at the beginning, it is necessary to answer on the question: What is a complex representation of the real connecting operators?

10. Complex representation of the connecting operators.

The Neifeld operators induce spinor analogues on the spinor space. In this case, every I -th subspace $\mathbb{R}_{(2,2)}^4 \subset R_{(n-2(I-1), n-2(I-1))}^{2n-4(I-1)} \oplus \mathbb{C}^{2(I-1)}$ has its own Neifeld operator m_I ($N = 2^{n/2-1}$, $I = \overline{1, \frac{n}{2}}$) [3, Algorithm 6.1, p. 77(eng),

pp. 154-155(31-32)(rus)], [2, pp. 31-32]. The Norden affinors Δ_I induce the two pairs of operators $(\tilde{\Delta}_I)_\pm, (\tilde{\tilde{\Delta}}_I)_\pm$ on the spinor space $(\Lambda, \dots = \overline{1, 2n-2I}, \tilde{\Lambda}, \dots = \overline{1, 2n-2(I-1)}, A, B, C, D \dots = \overline{1, \frac{2N^2}{2I}}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \dots = \overline{1, \frac{2N^2}{2I-1}})$

$$\begin{aligned} (\Delta_I)_{\tilde{\Lambda}}^{\tilde{\Psi}} \eta_{\tilde{\Psi}}^{\tilde{A}\tilde{B}} &= \eta_{\tilde{\Lambda}}^{\tilde{C}\tilde{D}} (\tilde{\Delta}_I)_{-\tilde{C}}^{\tilde{A}} (\tilde{\tilde{\Delta}}_I)_{-\tilde{D}}^{\tilde{B}} + \eta_{\tilde{\Lambda}}^{\tilde{C}\tilde{D}} (\tilde{\Delta}_I)_{+\tilde{C}}^{\tilde{A}} (\tilde{\tilde{\Delta}}_I)_{+\tilde{D}}^{\tilde{B}}, \\ (m_I)_\Lambda^{\tilde{\Psi}} (m_I)^\Lambda_{\tilde{\Lambda}} &= (\Delta_I)_{\tilde{\Lambda}}^{\tilde{\Psi}}, \\ (\tilde{m}_I)_{\pm\Lambda}^{\tilde{\Psi}} (\tilde{m}_I)^\Lambda_{\pm\tilde{\Lambda}} &= (\tilde{\Delta}_I)_{\pm\tilde{\Lambda}}^{\tilde{\Psi}}, \quad (\tilde{\tilde{m}}_I)_{\pm\Lambda}^{\tilde{\Psi}} (\tilde{\tilde{m}}_I)^\Lambda_{\pm\tilde{\Lambda}} = (\tilde{\tilde{\Delta}}_I)_{\pm\tilde{\Lambda}}^{\tilde{\Psi}}. \end{aligned}$$

Note that Δ_I is both the identity operator on $\mathbb{R}_{(n-2I, n-2I)}^{2n-4I} \oplus \mathbb{C}^{2(I-1)}$ and the canonical Norden affinor on $\mathbb{R}_{(2,2)}^4$. Let z_I be equal to 0 for $\ll - \gg$ and 1 for $\ll + \gg$ then one can define the pair of operators $\tilde{M}_K, \tilde{\tilde{M}}_K$ ($A, B, C, D \dots = \overline{1, N}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \dots = \overline{1, 2N^2}, K, J = \overline{1, 2N}, K \neq J$)

$$\begin{aligned} \tilde{M}_K &:= \tilde{m}_{\frac{n}{2}z_{\frac{n}{2}}} \tilde{m}_{\frac{n}{2}-1z_{\frac{n}{2}-1}} \dots \tilde{m}_{2z_2} \tilde{m}_{1z_1}, & \tilde{\tilde{M}}_K &:= \tilde{\tilde{m}}_{\frac{n}{2}z_{\frac{n}{2}}} \tilde{\tilde{m}}_{\frac{n}{2}-1z_{\frac{n}{2}-1}} \dots \tilde{\tilde{m}}_{2z_2} \tilde{\tilde{m}}_{1z_1}, \\ (\tilde{M}_K)_C^{\tilde{A}} (\tilde{M}_K^*)_{\tilde{A}}^B &= \delta_C^B, & (\tilde{\tilde{M}}_K)_C^{\tilde{A}} (\tilde{\tilde{M}}_K^*)_{\tilde{A}}^B &= \delta_C^B, \\ (\tilde{M}_K)_C^{\tilde{A}} (\tilde{M}_J^*)_{\tilde{A}}^B &= 0, & (\tilde{\tilde{M}}_K)_C^{\tilde{A}} (\tilde{\tilde{M}}_J^*)_{\tilde{A}}^B &= 0, \\ \sum_{K=1}^{2N} (\tilde{M}_K)_A^{\tilde{C}} (\tilde{M}_K^*)_{\tilde{B}}^A &= \delta_{\tilde{B}}^{\tilde{C}}, & \sum_{K=1}^{2N} (\tilde{\tilde{M}}_K)_A^{\tilde{C}} (\tilde{\tilde{M}}_K^*)_{\tilde{B}}^A &= \delta_{\tilde{B}}^{\tilde{C}} \end{aligned}$$

for the spinor space. Finally, one can obtain the $2N$ complex representations of the connecting operators $(\eta_K)_\Lambda^{AB} := (\tilde{M}_K^*)_{\tilde{A}}^A (m_\Lambda^{\tilde{A}} \eta_{\tilde{\Lambda}}^{\tilde{A}\tilde{B}}) (\tilde{M}_K^*)_{\tilde{B}}^B$ ($\Lambda, \dots = \overline{1, n}, \tilde{\Lambda}, \dots = \overline{1, 2n}$), but not all of them can be significant. Note that $I_{\tilde{\Lambda}}^{\tilde{\Omega}}(f_I)_{\tilde{\Omega}}^{\tilde{\Psi}}$ ($\tilde{\Lambda}, \dots = \overline{1, 2n-2(I-1)}$) is the complex pseudo-orthogonal transformation, where $(f_I)_{\tilde{\Lambda}}^{\tilde{\Psi}}$ is both the complex structure on $\mathbb{R}_{(2,2)}^4$ and the identity operator on $\mathbb{R}_{(n-2I, n-2I)}^{2n-4I} \oplus \mathbb{C}^{2(I-1)}$ in the definition of the Norden affinor $(\Delta_I)_{\tilde{\Lambda}}^{\tilde{\Psi}} = \frac{1}{2}(\delta_{\tilde{\Lambda}}^{\tilde{\Psi}} + I_{\tilde{\Lambda}}^{\tilde{\Omega}}(f_I)_{\tilde{\Omega}}^{\tilde{\Psi}})$, where $I_{\tilde{\Lambda}}^{\tilde{\Omega}}|_{\mathbb{R}_{(n-2I, n-2I)}^{2n-4I} \oplus \mathbb{C}^{2(I-1)}} = \delta_{\tilde{\Lambda}}^{\tilde{\Omega}}|_{\mathbb{R}_{(n-2I, n-2I)}^{2n-4I} \oplus \mathbb{C}^{2(I-1)}}$, $I_{\tilde{\Lambda}}^{\tilde{\Omega}}|_{\mathbb{R}_{(2,2)}^4} = i\delta_{\tilde{\Lambda}}^{\tilde{\Omega}}|_{\mathbb{R}_{(2,2)}^4}$. So it is necessary to answer on the question to justify the spinor decomposition of the Norden affinor: How to construct spinor analogs of complex orthogonal transformations? All significant complex representations are equivalent in all its properties, and therefore K will be omitted if the number of a considered representation is not significant.

11. Complex orthogonal transformations. Involutions.

More than once, we will be faced with theorems named after Élie Cartan in connection with spinors. It is one of them. The theorem was generalized to quadratic forms over arbitrary fields by Dieudonné.

Theorem 1. (*Cartan-Dieudonné*)[12, p. 33(rus)] *Let E be a vector space of dimension $n \geq 1$. Every isometry $f \in O(g)$ which is not the identity is the composition of at most n reflections with respect to hyperplanes, where $n = \dim E$ and g is a nondegenerate quadratic form on E .*

In 1938, Cartan was published this result in [15, Chapter I, Section 10]. Thus, any complex orthogonal transformation S_Λ^Ψ ($\Lambda, \Psi, \dots = \overline{1, n}, A, B, \dots = \overline{1, N}$)

can be represented as a product of elementary transformations of the form $\pm(r_\Lambda r^\Psi - \delta_\Lambda^\Psi)$. The elementary transformations can be decomposed with the help of the triple product for the connecting operators as $(r_\Lambda r^\Psi - \delta_\Lambda^\Psi)\eta_\Psi^{AB} = \eta_{\Lambda CD}(r^\Omega \eta_\Omega^{CB})(r^\Psi \eta_\Psi^{AD})$ [21, pp. 272-275]. Then on the spinor space, spinor analogs from the $Spin(g, n)$ for special orthogonal transformations are induced [3, p. 28, eq. (6.42)(eng), p. 156(33), eq. (6.42)(rus)], [2, p. 33, eq. (6.42)].

$$S_\Lambda^\Psi \eta_\Psi^{AB} = \eta_\Lambda^{CD} \tilde{S}_C^A \tilde{\tilde{S}}_D^B.$$

In the case when the algebra identity preserved, $S_C^A := \tilde{S}_C^A = \tilde{\tilde{S}}_C^A$. For the octonions, the automorphism group keeps the algebra identity, therefore the controlling spin-tensor must be invariant under spinor orthogonal transformations corresponding to automorphism group transformations that reduces the dimension on 7 else. In more detail, the block-diagonal structure can be found in [28, p. 94-109]. For the non-special orthogonal transformation the decomposition

$$S_\Lambda^\Psi \eta_\Psi^{BA} = \eta_{\Lambda CD} \tilde{S}^{CA} \tilde{\tilde{S}}^{DB}$$

is performed. This makes it possible to pass to infinitesimal transformations by the equation [3, p. 52, eq. (10.6)(eng), p. 184(61), eq. (10.6)(rus)], [2, p. 61, eq. (10.6)].

$$T_C^A = \frac{1}{2} T^{\Theta\Phi} \eta_\Phi^{AB} \eta_{\Theta CB}.$$

For $n=8$, it is easy to calculate the dimension of the automorphism group for alternative-elastic algebras using this equation [7]. For the involution, for the given signature, only one of the two decompositions

$$S_\Lambda^{\Psi'} \bar{\eta}_{\Psi'}^{A'B'} = \eta_\Lambda^{CD} \tilde{S}_C^{A'} \tilde{\tilde{S}}_D^{B'}, \quad S_\Lambda^{\Psi'} \bar{\eta}_{\Psi'}^{B'A'} = \eta_{\Lambda CD} \tilde{S}^{CA'} \tilde{\tilde{S}}^{DB'}$$

is executed [3, p. 29, eq. (6.47)(eng), p. 156(33), eq. (6.47)(rus)], [2, p. 33, eq. (6.47)].

12. Klein-Fock-Gordon equation.

In the first time (1926), the Klein-Fock-Gordon equation $(\square - m^2)\phi = 0$ was considered by Schrodinger, but was submitted by Vladimir Fock in 1927. In 1928, British physicist Paul Dirac formulated the equation, the simplified record of which has the form $(i\gamma^i \partial_i - m)\phi = 0$ ($i = \overline{1,4}$). In order to get the Klein-Fock-Gordon equation, the Dirac equation is necessary multiplied by $(i\gamma^i \partial_i + m)$ that will lead to the Clifford equation for $\mathbb{R}_{(1,3)}^4$. This initiated the study of Clifford equation solutions for $n = 4$ [25]. However, it is more convenient to consider the inclusion $\mathbb{R}_{(1,3)}^4 \rightarrow \mathbb{C}^4$ and to pass to the spinor formalism for the complex space \mathbb{C}^4 and the real representation $\mathbb{R}_{(4,4)}^8$. Why? Because there is an universal geometric construction: the Cartan triality principle (in 1925) [15, p. 175(rus)], [28, p. 534]. In the future, one can always go back with the help of the appropriate real inclusion using the reverse motion. For example, for the inclusion $H_i^\Lambda : \mathbb{R}_{(1,3)}^4 \rightarrow \mathbb{C}^4$, one can define the Infeld-van der Waerden symbols $g_i^{AA'} := H_i^\Lambda \eta_\Lambda^{AB} S_B^{A'}$ ($i, j, \dots = \overline{0,3}$,

$\Lambda = \overline{0, 3}$, $A, B, A', B' \dots = \overline{1, 2}$). By this equation, the involution $S_B^{A'}$ is introduced in the definition of the symbols $g_i^{AA'}$. Accordingly, the involution $S_A^{A'}$ will generate the involution \bar{S} for the complex Dirac operators $\bar{\gamma}_i = \bar{S}\gamma_i\bar{S}$, and in this special basis, $||S|| = ||\gamma^2||$ will be executed. Thus, we can define the Majorana spinor as $\psi^i = \gamma^2\bar{\psi}^i$ which corresponds to the presentation [36, p. 98, Appendix E (eng)], [3, p. 89, Section 14.7.1(eng), p. 225(102), Section 14.5.6(rus)], [2, p. 102, Section 14.5.6].

13. Cartan triality principle.

Theorem 2. *(The triality principle for two B-cylinders)[3, p. 91, Section 14.7.2(eng), p. 227(104), Section 14.5.7(rus)], [2, p. 104, Section 14.5.7], [4, p. 70, Section 5.7(eng), p. 183(94), Section 5.7(rus)], [8]. In the projective space \mathbb{CP}_7 , there are two quadrics (two B-cylinders) with the following main properties:*

1. *The planar generator \mathbb{CP}_3 of a one quadric will define one-to-one the point R on the other quadric.*
 2. *The planar generator \mathbb{CP}_2 of a one quadric will uniquely define the point R on the other quadric. But the point R of the second quadric can be associated to the manifold of planar generators \mathbb{CP}_2 belonging to the same planar generator \mathbb{CP}_3 of the first quadric.*
 3. *The rectilinear generator \mathbb{CP}_1 of a one quadric will define one-to-one the rectilinear generator \mathbb{CP}_1 of the other quadric. And all the rectilinear generators belonging to the same planar generator \mathbb{CP}_3 of the first quadric define the beam centered at R belonging to the second quadric.*
- For even $n < 8$ the dimension of the base space is greater than the one of the spinor space.
 - For $n = 8$ the dimension of the base space is equal to the one of the spinor space.
 - For even $n > 8$ the dimension of the base space is less than the one of the spinor space.

Thus, for $n \leq 8$, one can construct the division normalized algebra. For $n > 8$, the 0-divisors appear, because the inclusion of the base space in the spinor one is performed with the help of the inclusion operator $P^i_B := \eta^i_{AB}X^A$, which is included in the definition of algebraic basis elements and ultimately determines the 0-divisors of an algebra itself. The weakly normalization identity has the form $g_{kr}\eta_{[ij]}^k\eta_{lm}^r = g_{kr}\eta_{j(i}\eta_{r|m}^{r|l)}$. This identity is the normalization identity for $n=8$ only ($g_{kr}\eta_{j(i}\eta_{r|m}^{r|l)} = g_{jm}g_{il}$ in this case). What is based the triality principle on?

14. Rosenfeld null-pairs.

Rosenfeld null-pairs lie in the heart of the triality principle. Let the pair $X^A := (X^b, Y_a)$ ($a, b, \dots = \overline{1, 4}$, $A, B, \dots = \overline{1, 8}$) be *Rosenfeld null-pair* [27], [29, p. 378]. In the space $\mathbb{CP}^4 = {}'\mathbb{C}^4/{}'\mathbb{C}$ (where ${}'\mathbb{C}^n = \mathbb{C}^n/0$), X^b defines the

point and Y_a defines the plane with *incidence condition* $X^b Y_a = 0$ [29, p. 362]. Therefore, we can construct the space $\mathbb{C}\Pi^4 = \mathbb{C}^{*4}/\mathbb{C}$, which is *dual space* to \mathbb{CP}^4 . Then the space $\mathbb{CP}^4 \times \mathbb{C}\Pi^4$ is *Rosenfeld null-pair space*. It should be noted that such the spaces were studied by Sintcov [31] and Kotelnikov [20] for the first time. Both a twistor is constructed from pair spinors and a bitwistor consists of two twistors, and an isotropic bitwistor is the null-pair with the incidence condition. The isotropic bitwistor X^A is associated to the parity $X^a = i r^{ab} Y_b$, $r^{ab} = -r^{ba} := r^\alpha \eta_\alpha^{ab}$ ($\alpha, \dots = \overline{1, 6}$) as well as an twistor is associated to pair spinors [25, v. 2, p. 47, eq. (6.1.13)], where r^{ab} is the same as in [25, v. 2, p. 306, eq. (9.3.7)] for the real inclusion in the special basis. A vector of the base space is determined by analogy with [25, v. 2, p. 306-307] just as [3, p. 83-84(eng), p. 218-223(95-100)(rus)]. The metric tensor $g_{\Lambda\Psi}$ ($\Lambda, \Psi, \dots = \overline{1, 8}$) acts on the base space, and its metric analog ε_{AB} acts on the bitwistor space. Thus, the two quadrics are define:

- the quadric, determined with the help of $\varepsilon_{AB} X^A X^B = 0$;
- the quadric, determined by the 6 parameters r^{ab} with the help of $g_{\Lambda\Psi} r^\Lambda r^\Psi = 0$.

This quadrics are the B-cylinders from Theorem 2. This allows to obtain the representation of an isotropic twistor on the isotropic cone of the base space $R_{(2,4)}^6$. This is completely analogous to the representation of a spinor on the isotropic cone of the space-time with only the one difference: the dimensions of the flag and the flagpole are increased by 1 [3, p. 72-77(eng), p. 205-210(82-87)(rus)]. In order to construct the representation, one can need to use the result of the paper [35] and to define the matrix differential. Besides, the Theorem 2 combines both the Cartan triality principle and the Klein correspondence [25, v. 2, p. 307-313].

15. Spinor formalism. Partial solutions of the Clifford equation.

Solutions of the Clifford equation for $n=4$ are given in [25] and associated to the Pauli matrixes. As is known, they are 4: an one antisymmetric matrix and three symmetric matrixes. In Russia, for $n=6$, the first appearance of partial solutions of the Clifford equation seems to be attributed to [24] in 1959. They are related to Plucker coordinates for the Grassmannian $G(2, 4)$. To foreign publications, the article [18],[13] can be attributed. Further development of these results were obtained by Ukrainian physicist Yu. P. Stepanpovskiĭ [32], [33]. Thus, for $n=6$, the partial solutions of the Clifford equation can be represented with the help of six antisymmetric matrixes. It should be noted that the specified formalism is closely connected with the Bogolyubov-Valatin transformations [30]. It can be shown that for $n=8$, partial solutions of the Clifford equation can be represented with the help of 8 matrixes: seven antisymmetric matrixes and an one symmetric matrix [5], [3, p. 77, Section 14.4(eng), p. 211(88), Section 14.4 (rus)], [2, p. 88, Section 14.4]. Therefore, we must assume that for even n , partial solutions of the Clifford equation can be represented with the help of n matrixes: \tilde{q} antisymmetric matrixes and q symmetric matrixes ($\tilde{q} + q = n$). Note, that the Penrose spinor formalism for $n=4$ is based on the Klein correspondence [25], while the spinor formalism for

even n can be considered on the base of the Cartan triality principle [9], [2]-[4]. And because the Klein correspondence is a particular case of the Cartan triality principle then it makes sense to consider the spinor formalism for $n=8$ and then to go back to $n=4$ with the help of the reverse motion.

16. Bott periodicity. Classification of metric group alternative-elastic algebras.

In 1908, Cartan discovered the principle of periodicity for the Clifford algebras [16], [11], [21] $CL(g_{(n+8,0)}^{n+8}) \cong CL(g_{(n,0)}^n) \otimes \mathbb{R}$ [16]. But this property is true for any signature of the metric. This means that the main properties must be preserved for algebras with the same value $n \bmod 8$. The presence of an symmetric metric spinor on the spinor space provides the existence of the group structure with respect to multiplication for $n \bmod 8 = 0$. This implies that since octonions are endowed with the group alternative-elastic structure then alternative-elastic algebras exist for any $n \bmod 8 = 0$. Moreover, octonions is initial induction step for the construction of such the algebras for $n \bmod 8 = 0$. To construct the next even-dimensional connecting operators, one can use the inductive transition of the form

$$\begin{aligned} \eta_{\Lambda}^{AB} &= \begin{pmatrix} \eta_{\alpha}^{ab} & \frac{1}{2}(i\eta_{n-1} + \eta_n)\delta^a_d \\ \frac{1}{2}(-i\eta_{n-1} + \eta_n)\delta_c^b & -(\eta^T)_{\alpha cd} \end{pmatrix}, \\ \sigma_{\Lambda}^{AB} &= \begin{pmatrix} (\eta^T)_{\alpha ab} & \frac{1}{2}(i\eta_{n-1} + \eta_n)\delta_a^d \\ \frac{1}{2}(-i\eta_{n-1} + \eta_n)\delta_c^b & -\eta_{\alpha}^{cd} \end{pmatrix} \end{aligned}$$

with different variations of this form as shown in Algorithm 9.1 [3, p. 50(eng), p. 181(58)(rus)], [2, p. 58], in Algorithm 1 [6]; $\Lambda = \overline{1, n}$; $\alpha = \overline{1, n-2}$; $a, b, \dots = \overline{1, N/2}$; $A, B, C, D, K = \overline{1, N}$; $N := 2^{n/2-1}$. Note, that similar constructions are considered in [28, p. 518] for alternions. Besides, such the transition is given in [25, p. 65, eq. (6.2.18)]. Thus, the connecting operation can be constructed inductively for any $n \bmod 8 = 0$. For inclusion $H_i^{\Lambda}: \mathbb{R}^n \rightarrow \mathbb{C}^n$, $(i, j, \dots = \overline{1, n})$, any metric alternative-elastic algebra can be expanded into the basis, the I-element of which is the algebra of the form

$$(\eta_I)_{ij}^k := \sqrt{2}\eta_i^{AB}\eta_{jCA}\eta_{DB}^k((X_I)^C(X_I)^D + (1 - \frac{1}{2}(X_I)^K(X_I)_K)\frac{2}{N}\varepsilon^{CD}),$$

where ε^{CD} is the metric spinor on the spinor space, and $(1 - \frac{1}{2}(X_I)^K(X_I)_K)\frac{2}{N}\varepsilon^{CD}$ guarantee the existence of the algebra identity. We will consider such the orthogonal basis elements $(X_I)^C$, which are transformed into each other with the help of the orthogonal transformation only and $(X_I)^C(X_I)_C := 2$. Indeed, any controlling symmetric spinor θ^{CD} can be decomposed as

$$\theta^{CD} = \sum_{I=1}^J \alpha_I \underbrace{(X_I)^C(X_I)^D}_{:=(\theta_I)^{CD}} + \underbrace{(1 - \sum_{I=1}^J \alpha_I)}_{:=\alpha_0} \underbrace{\frac{2}{N}\varepsilon^{CD}}_{(\theta_0)^{CD}}.$$

In particular, Cayley-Dickson algebras admit such the decomposition and are special orthogonal metric alternative-elastic algebras. This means, that such

the algebra is generated by special orthogonal transformations from the single basis element. This element is generated from the controlling spinor with the significant coordinates $X^1 := 1$, $X^{2^{\frac{n}{2}-2}+1} := 1$ [6, p. 8]. The construction of the sedenion algebra, based on this expansion, is realized in Appendix [6]. One can easily classify these algebras for $n = 8$. This classification is based on the one of eigenvalues of the controlling spin-tensor θ^{CD} [7]. For the octonion algebra [7, ex. 1], θ^{CD} has the single significant eigenvalue, and therefore the dimension is reduced on 7 else. A similar classification can be constructed for any $n \bmod 8 = 0$.

17. Complex and real representations of (pseudo-)Riemannian spaces.

Under *complex analytical Riemannian space* $\mathbb{C}V_n$, we will further understand an analytical complex manifold supplied with *analytical quadratic metric*, i.e., a metric defined by means of a symmetric nonsingular tensor $g_{\Lambda\Psi}$ (here $\Lambda, \Psi, \dots = \overline{1, n}$; $\tilde{\Lambda}, \tilde{\Psi}, \dots = \overline{1, 2n}$), which the coordinates are analytical functions of the point coordinates. To this tensor, there corresponds *complex Riemannian torsion-free connection*, the coefficients of which are defined by the Christoffel symbols, and hence these coefficients are analytical functions. The tangent bundle to this manifold $\tau^{\mathbb{C}}(\mathbb{C}V_n)$ has fibers $\tau_x^{\mathbb{C}} \cong \mathbb{C}\mathbb{R}^n$ that is fibers isomorphic to the n -dimensional complex Euclidian space, the metric of which is defined by the value of the metric tensor at the given point. The real representation V_{2n} of $\mathbb{C}V_n$ has the tangent bundle $\tau^{\mathbb{R}}(V_{2n})$ with fibers isomorphic to $\mathbb{R}_{(n,n)}^{2n}$. Let an atlas $(U; x^{\tilde{\Lambda}})$ be set on V_{2n} . We will consider *reparametrization* of this atlas $(U; w^{\Lambda})$ such that $w^{\Lambda} = \frac{1}{\sqrt{2}}(u^{\Lambda}(x^{\tilde{\Lambda}}) + iv^{\Lambda}(x^{\tilde{\Lambda}}))$, which is locally solvable as $x^{\tilde{\Lambda}} = x^{\tilde{\Lambda}}(u^{\Lambda}, v^{\Lambda})$. Set

$$m^{\Lambda}_{\tilde{\Lambda}} := \frac{1}{\sqrt{2}}\left(\frac{\partial u^{\Lambda}}{\partial x^{\tilde{\Lambda}}} + i\frac{\partial v^{\Lambda}}{\partial x^{\tilde{\Lambda}}}\right) =: \frac{\partial w^{\Lambda}}{\partial x^{\tilde{\Lambda}}}, \quad m_{\Lambda}^{\tilde{\Lambda}} := \frac{1}{\sqrt{2}}\left(\frac{\partial x^{\tilde{\Lambda}}}{\partial u^{\Lambda}} - i\frac{\partial x^{\tilde{\Lambda}}}{\partial v^{\Lambda}}\right) =: \frac{\partial x^{\tilde{\Lambda}}}{\partial w^{\Lambda}},$$

$$f_{\tilde{\Lambda}}^{\tilde{\Psi}} := \frac{\partial v^{\Lambda}}{\partial x^{\tilde{\Lambda}}} \frac{\partial x^{\tilde{\Psi}}}{\partial u^{\Lambda}} - \frac{\partial u^{\Lambda}}{\partial x^{\tilde{\Lambda}}} \frac{\partial x^{\tilde{\Psi}}}{\partial v^{\Lambda}}.$$

Then the operators $m^{\Lambda}_{\tilde{\Lambda}}$ are coincide with the ones from Section 8 [3, p. 9, Section 3(eng), p. 133(10), Section 3(rus)], [2, p. 10, Section 3].

18. Complex and real connections over Riemannian spaces. Case I.

Therefore, if one can define the connection $\nabla_{\tilde{\Phi}} G_{\tilde{\Lambda}\tilde{\Psi}} = 0$ ($\tilde{\Lambda}, \tilde{\Psi}, \dots = \overline{1, 2n}$) in the tangent bundle for the real representation then the connection $\nabla_{\Phi} g_{\Lambda\Psi} = 0$, $\bar{\nabla}_{\Phi} \bar{g}_{\Lambda'\Psi'} = 0$ ($\Lambda, \Psi, \dots = \overline{1, n}$) is induced in the tangent bundle for the complex representation by the covariant constancy of the Neifeld operators. Then onto the spinor bundle with fibers isomorphic \mathbb{C}^N , the connection prolongs with the help of the covariant constancy of the connecting operators. If someone can want to move into the real inclusion then he must request the covariant constancy of the inclusion operator that will lead to the Riemannian connection $\nabla_i g_{jk} = 0$ ($i, j, \dots = \overline{1, n}$) [3, p. 59, Corollary 11.1(eng), p. 190(67), Corollary 11.1(rus)], [2, p. 67, Corollary 11.1], [4, Section 3]. In the last case, on the spinor bundle, the induced involution must be the covariant constant.

On the spinor space, such the connection is called *connection compatible with involution*. Hereinafter, all connections are torsion-free.

19. Revers motion: Lie operator analogues. Spin-pair space. Case II.

The connection can be prolonged an another way onto the spinor bundle [3, p.59, Theorem 11.2(eng), p. 191(68), Theorem 11.2(rus)], [2, p. 68, Theorem 11.2]. On a tangent fiber of the real representation V_{2n} , one can define the operator $P_{\tilde{A}}^{\tilde{\Psi}} := \eta_{\tilde{B}\tilde{A}}^{\tilde{\Psi}} X^{\tilde{B}}$ and $P_{\tilde{\Psi}}^{\tilde{A}} := \eta_{\tilde{\Psi}}^{\tilde{B}\tilde{A}} Y_{\tilde{B}}$ ($\tilde{A}, \tilde{\Psi}, \dots = \overline{1, 2n}, \tilde{A}, \tilde{B}, \dots = \overline{1, 2N^2}$) such that $X^{\tilde{A}} Y_{\tilde{A}} = 2$ and $P_{\tilde{A}}^{\tilde{A}} P_{\tilde{\Psi}}^{\tilde{\Psi}} = G_{\tilde{A}\tilde{\Psi}}$. This is always possible for $n \geq 8$. Then the compatibility condition of the connections for the real representation will have the form $\nabla_{\tilde{A}} P_{\tilde{\Psi}}^{\tilde{A}} = 0$. For the complex representation, the operators $(P_K)^{\Psi}_A := m_{\tilde{\Psi}}^{\Psi} P_{\tilde{A}}^{\tilde{\Psi}} (\tilde{M}_K)_A^{\tilde{A}}$, $(P_K)^{\Psi'}_A := \bar{m}_{\tilde{\Psi}}^{\Psi'} P_{\tilde{A}}^{\tilde{\Psi}} (\tilde{M}_K)_A^{\tilde{A}}$, $(P_K^*)^{\Psi}_A := m_{\tilde{\Psi}}^{\Psi} P_{\tilde{A}}^{\tilde{\Psi}} (\tilde{M}_K^*)_A^{\tilde{A}}$, $(P_K^*)^{\Psi'}_A := \bar{m}_{\tilde{\Psi}}^{\Psi'} P_{\tilde{A}}^{\tilde{\Psi}} (\tilde{M}_K^*)_A^{\tilde{A}}$ ($\Lambda, \Psi, \dots = \overline{1, n}, A, B = \overline{1, N}, K = \overline{1, 2N}$) are defined. Then for the K-th complex representation, the Riemannian connection determines by the covariant constancy of the Neifeld operators and the operators $\tilde{M}_K, \tilde{M}_K^*$. This induces the prolongation of the connection onto the complex spinor bundle by the covariant constancy of the operators P_K, P_K^* . Analogically, for the real inclusion, it is necessary to demand the covariant constancy of the inclusion operator H . If someone demands not only the covariant constancy but also the ordinary one then he can define the Lie operator analogues on the spinor bundle [3, p.61, Theorem 11.3(eng), p. 192(69), Theorem 11.3(rus)], [2, p. 69, Theorem 11.3]

$$\begin{aligned} L_x(Y_K)^A &:= x^\Omega \partial_\Omega (Y_K)^A - \sum_{J=1}^{2N} (Y_J)^B (P_J)^\Psi_B (P_K^*)_{\Omega}^A \partial_\Psi x^\Omega, \\ \bar{L}_{\bar{x}}(\bar{Y}_K)^A &:= \bar{x}^{\Omega'} \bar{\partial}_{\Omega'} (\bar{Y}_K)^A - \sum_{J=1}^{2N} (\bar{Y}_J)^B (\bar{P}_J)^{\Psi'}_B (\bar{P}_K^*)_{\Omega'}^A \bar{\partial}_{\Psi'} \bar{x}^{\Omega'}. \end{aligned}$$

For the inclusion $\mathbb{R}_{(1,3)}^4 \rightarrow \mathbb{C}^4$ ($n=4$), the constructed operators have the form [3, p.96, Example 14.2(eng), p. 262, Addition (rus)]

$$\begin{aligned} (P_K^*)_i^A &:= \frac{1+i}{4} \begin{pmatrix} 0 & p_1 - p_2 \\ 0 & 1 - p_1 p_2 \\ p_1 - p_2 & 0 \\ i(1 - p_1 p_2) & 0 \end{pmatrix}, \\ (P_K)_i^A &:= \frac{1-i}{4} \begin{pmatrix} 0 & p_1 - p_2 \\ 0 & 1 - p_1 p_2 \\ p_1 - p_2 & 0 \\ -i(1 - p_1 p_2) & 0 \end{pmatrix}, \end{aligned}$$

where $p_1 := \pm 1, p_2 := \pm 1$. Therefore, one can construct the one-to-one mapping $(a, b, \dots = \overline{1, 4}, A, B, \dots = \overline{1, 2}, i, j, \dots = \overline{1, 4})$

$$x^i \cdot \underbrace{((P_2^*)_i^A, (P_3^*)_i^B)}_{:= P_i^a} = \underbrace{((X_2)^A, (X_3)^B)}_{:= X^a}, \quad x^i \cdot \underbrace{((P_2)_{iA}, (P_3)_{iB})}_{:= P_{ia}} = \underbrace{((X_2^*)_A, (X_3^*)_B)}_{:= X_a}.$$

Then on the spin-pair $X^a = ((X_2)^A, (X_3)^B)$, the metric and the involution of the form

$$\varepsilon_{ab} = \begin{pmatrix} 0 & (P_2)^i{}_A (P_3)_{iB} \\ (P_3)^i{}_C (P_2)_{iD} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$S_a{}^{b'} = \begin{pmatrix} (P_2)^i{}_A (\bar{P}_2^*)_{iC'} & (P_2)^i{}_A (\bar{P}_3^*)_{iD'} \\ (P_3)^i{}_B (\bar{P}_2^*)_{iC'} & (P_3)^i{}_B (\bar{P}_3^*)_{iD'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

are defined. Therefor, it is possible to construct *Lie pair-spin operators*

$$L_x(Y)^a = x^i \nabla_i Y^a - P^j{}_b Y^b P_i{}^a \nabla_j x^i, \quad L_X(Y)^a = X^c \nabla_c Y^a - Y^c \nabla_c X^a.$$

Note, that the expansion $x^i := (P_2)^i{}_A (X_2)^A + (P_3)^i{}_A (X_3)^A$ is an one of the vector into the two isotropic vectors. And for the isotropic vectors, *Lie spinor operators* were constructed in [25, v. 2, pp. 101-103 (eng)]. The restricted Lorentz transformation converts the vector x^i to the vector \tilde{x}^i . This induces the spinor transformation: $(X_K)^A \mapsto (\tilde{X}_K)^A$. The spinor transformations act on the operators $(P_K^*)_{iA}$, $(P_K)^i{}_A$, more precisely, on the controlling spinor $X^{\tilde{A}}$ appearing in the definition of these operators. In turn, this induces the spin-pair transformation of the spin-pair $X^a = ((X_2)^A, (X_3)^B) \mapsto \tilde{X}^a$.

20. Normalization of Grassmannians. Twistor equation. Case III.

Normalization of the complex Grassmannian $G_{\mathbb{C}}(2N, 4N)$ [23] is an analytical differential map N of a domain $D \subset G_{\mathbb{C}}(2N, 4N)$ into the dual Grassmannian $G_{\mathbb{C}}^*(2N, 4N)$. $(\mathbb{C}^*)^{2N}(Y_{\tilde{A}})$ ($\tilde{A}, \tilde{B}, \dots = \overline{1, 2N}$) is called *normalizative subspace for subspace* $\mathbb{C}^{2N}(X_{\tilde{B}})$. The last must not have common directions with the first. $Y_{\tilde{A}}, X_{\tilde{B}}$ are basis elements of the corresponding subspace. In local coordinates, the correspondence N must be given with the help of the parametric equations $X_{\tilde{B}} = X_{\tilde{B}}(r^A)$, $Y_{\tilde{A}} = Y_{\tilde{A}}(r^A)$ ($A, \Psi, \dots = \overline{1, n}$). The boundary points of D , if they exist and for which the plane \mathbb{C}^{2N} has an one common direction with $(\mathbb{C}^*)^{2N}$ at least, form *absolute of the normalization*. The derivational equations of the normalized Grassmannian can have the form $\nabla_{\Lambda} X_{\tilde{A}} = i\gamma_{\Lambda\tilde{A}}{}^{\tilde{B}} Y_{\tilde{B}}$, $\nabla_{\Lambda} Y_{\tilde{B}} = 0$ in the particular case. If γ_{Λ} are satisfied the Clifford equation then such the normalization is called *spinor normalization*, and the derivational equations [23, eq. (1.2)] can be reduced to $\nabla_{\Lambda} X^A = \underbrace{i\eta_{\Lambda}{}^{AB} Y_B}_{:= P_{\Lambda}{}^A}$, $Y_B = \text{const}$ ($A, B, \dots = \overline{1, N}$). They can be rewritten as

the twistor equation [25, v. 2, p. 463, eq. (B.94a)]

$$\eta_{\Lambda AB} \nabla_{\Psi} X^A + \eta_{\Psi AB} \nabla_{\Lambda} X^A = \frac{2}{n} g_{\Lambda\Psi} \eta^{\Phi}{}_{AB} \nabla_{\Phi} X^A.$$

For any X^A , the integrability condition is write down as $C_{\Lambda\Psi\Phi\Omega} = 0$ [3, Section 13], where $C_{\Lambda\Psi\Phi\Omega}$ is the Weyl tensor. Thus, the constructed connection is conformally Euclidean. For this case, the solution of the twistor equation has

the form $X^C := \dot{X}^C + iR^{CA}\dot{Y}_A$, $\dot{X}^C = const$, $\dot{Y}_A = const$, $R^{CA} := r^\Lambda \eta_\Lambda^{CA}$. Recall that there are the $2N$ units of such the complex representations. If someone will find the sum on K then he will obtain the Killing equation from the twistor equation [3, Section 13, p. 68(eng), p. 200(77)(rus)]. For $n=6$, the particular case is considered in [10].

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